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WEAKLY PURE SUBMODULES OF MULTIPLICATION MODULES

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Abstract

Let R be a commutative ring with non-zero identity and M be a unital R-module. Then R-submodule N of M is called weakly pure, if for every Boolean ideal I of R, $IN = N \cap IM$. This paper is devoted to investigate some of the properties of weakly pure submodules of multiplication modules.

1. Introduction

Throughout this paper, all rings will be commutative with non-zero identity and have at least one non-zero Boolean ideal and all modules will be unitary. Pure submodules of multiplication modules have been investigated by Ali and Smith [1] and others.

A submodule N of R-module M is called *pure*, if $IN = N \cap IM$, for every ideal I of R. The aim of this paper is to prove for weakly pure submodules some of the results given in [1] for pure submodules of multiplication modules.

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Now, we define the concepts that we will use. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. An R-module M is called a *multiplication module*, if for every submodule N of M, there exists an ideal I of R such that $N = IM$. In this case, I is called *presentation ideal of* N. If N is an R-submodule of multiplication R-module M, then $N = (N : M)M$. Also $N = ann(M / N)M$.

2. Main Results

Definition 1. Let M be a module over a ring R. A proper submodule N of M is said to be *prime* (weakly prime), if $rm \in N(0 \neq rm \in N)$ for $r \in R$ and $m \in M$ implies that either $m \in N$ or $r \in (N : M)$.

Definition 2. An ideal I of R is called *Boolean ideal*, if every element of I is idempotent.

Definition 3. An R-submodule N of R-module M is called *weakly pure*, if $IN = N \cap IM$, for every Boolean ideal I of R.

Definition 4. An R-module M is called *prime*, if $(0 : M) = (0 : N)$ for every submodule N of M.

Theorem 1. *If I is a Boolean ideal of R, then $IJ = I \cap J$ for every ideal J of R.*

Proof. Assume that I is a Boolean ideal of R and $r \in I \cap J$. Then $r = r^2 \in I$ and $r = r^2 \in J$. So $r = r^2 \in IJ$, also $IJ \subseteq I \cap J$, hence $IJ = I \cap J$, for every ideal J of R. ■

Theorem 2. *Let M be a prime multiplication faithful R-module and N be a proper weakly pure submodule of M. Then, the following hold*:

(i) *The ideal $(N : M)$ is Boolean.*

(ii) *The ideal $(N : M)$ is idempotent.*

Proof. (i) Assume that $r \in (N : M) = ann(M/N)$. Then, $r(m + N) = r^2(m + N) = N = (r - r^2)(m + N) = (r - r^2)N$. Because $(r - r^2)N \subseteq (r - r^2)(m + N) = N$, hence for every arbitrary $m \in M$ and $\acute{n} \in N$, there exists $n \in N$, such that $(r - r^2)\acute{n} = (r - r^2)(m + n)$. So $(r - r^2)(m + \H{n}) = 0$, for $\H{n} = n - \acute{n} \in N$. Then $(m + \H{n}) = 0$ or $(r - r^2) \in (0 : m + \H{n})$. But $(m + \H{n}) \neq 0$ (if not $m \in N$, but N is a proper submodule and $m \in M$ is arbitrary, so $(m + \H{n}) \neq 0$), hence $(r - r^2) \in (0 : m + \H{n})$. But M is prime and faithful, so $(r - r^2) \in (0 : m + \H{n}) = (0 : R(m + \H{n})) = (0 : M) = 0$, and it implies that $r = r^2$ and so $(N : M)$ is a Boolean ideal.

(ii) By Theorem 1, we have $(N : M)^2 = (N : M)(N : M) = (N : M) \cap (N : M) = (N : M)$, hence the ideal $(N : M)$ is idempotent. ■

Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$, for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [2, Theorem 3.4], the product of N and K is independent of presentation of N and K. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$. Now we have the following results:

Corollary 3. *Let M be a prime multiplication faithful R-module. Then every proper weakly pure submodule of M is idempotent and in this case, if N is a proper weakly pure submodule of M, then $N = (N : M)N$.*

Proof. Let N be a proper weakly pure submodule of M. Then by Theorem 2, $N^2 = (N : M)^2M = (N : M)M = N$. Also $N = (N : M)^2M = (N : M)N$. ■

Corollary 4. *Let M be a prime multiplication faithful R-module and $0 \neq N$ be a proper weakly pure submodule of M. Then $ann(N : M) = 0$.*

Proof. For every $x \in ann(N : M)$, we have $x(N : M) = 0$, hence $xN = x(N : M)N = 0$, so that $x \in annN = annM = 0$, hence $x = 0$, so $ann(N : M) = 0$. ■

Theorem 5. *Let M be a prime multiplication faithful R-module and $0 \neq N$ be a proper weakly pure submodule of M. Then N is weakly prime, if and only if it is a prime submodule of M.*

Proof. Because every prime submodule is weakly prime, it is enough to show that if N is weakly prime, then N is prime. Assume that $0 \neq N$ is a proper weakly prime submodule of M that is not prime. Then by [4, Theorem 9] and Theorem 2, we have $N = (N : M)M = (N : M)^2 M = (N : M)N = 0$, which is a contradiction. Thus N is prime. ■

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